

ON THE COHOMOLOGY OF RAPOPORT-ZINK SPACES OF HODGE TYPE

SERIN HONG

ABSTRACT. We prove the conjecture of Harris on the cohomology of Rapoport-Zink spaces for unramified reductive groups, under the “Hodge-Newton decomposibility” assumption. Our proof is based upon Mantovan’s previous result for EL/PEL cases, combined with the “EL realization” of our Rapoport-Zink spaces.

CONTENTS

1. Introduction	1
2. Notations and Preliminaries	4
3. p -divisible groups with G -structure	6
4. Rapoport-Zink spaces of Hodge type	10
5. Harris conjecture for Rapoport-Zink spaces of Hodge type	14
References	22

1. INTRODUCTION

Rapoport-Zink spaces are formal moduli spaces of p -divisible groups with additional structures induced by a connected reductive group G over \mathbb{Q}_p . These spaces satisfy many local analogues of the properties of Shimura varieties. For example, they come with rigid analytic towers over the generic fibre which are equipped with a Weil descent datum and the action of $G(\mathbb{Q}_p)$. The construction of such spaces were first given by Rapoport and Zink in [RZ96] for $G = \mathrm{GL}_n$ and for EL/PEL cases, and recently extended to general unramified cases by Kim in [Kim13].

It has been widely believed that the l -adic cohomology of Rapoport-Zink spaces provides a geometric realization of local Langlands correspondences. Along this line is a famous conjecture by Harris in [Har01], which states that the l -adic cohomology of non-basic Rapoport-Zink spaces can be realized as a parabolic induction of the l -adic cohomology of the corresponding basic spaces. For $G = \mathrm{GL}_n$, a special case of the conjecture was already known due to the work of Boyer in [Boy99], and played an important role in the proof of the local Langlands correspondence by Harris and Taylor in [HT01].

The primary purpose of this paper is to prove Harris’s conjecture for general reductive groups, under a certain decomposibility assumption regarding the Newton polygon and the Hodge polygon. A similar assumption is made in Mantovan’s beautiful paper

[Man08], where the conjecture is proved for many EL/PEL cases. Following Mantovan's strategy, we will construct an analogue of Rapoport-Zink spaces associated to a specified parabolic subgroup, and compare its l -adic cohomology with the l -adic cohomology of the other spaces. The construction of this new space will be based on the local embedding of G into a group of EL type, constructed by the author in [Hong16].

For explanation of our result in more detail, we need some notations. We take $k = \bar{\mathbb{F}}_p$, W the ring of Witt vectors over $\bar{\mathbb{F}}_p$, and $K_0 = W[1/p]$. We will assume that our connected reductive group G over \mathbb{Q}_p is unramified, i.e., it is quasi-split and split over a finite unramified extension of \mathbb{Q}_p . This allows us to choose a \mathbb{Z}_p -model of G , which will be also denoted by G . We also fix an embedding $G \hookrightarrow \mathrm{GL}(\Lambda)$ for some fixed finite free \mathbb{Z}_p -module Λ .

Given an element $b \in G(K_0)$, we take X to be the p -divisible group corresponding to the Dieudonné module $M := \Lambda \otimes_{\mathbb{Z}_p} W$ with the Frobenius map $F = b \circ (1 \otimes \sigma)$. Our group G gives a finite family of F -invariant tensors $(t_i)_{i \in I}$ on M (see 3.1.1 for details). We will refer the pair $\underline{X} = (X, (t_i))$ as a p -divisible group with G -structure. Such an object comes with a generalized notion of the Newton polygon and the $(\sigma$ -invariant) Hodge polygon.

In our study, we will prove Harris's conjecture under the assumption that \underline{X} is of *Hodge-Newton type*. This assumption implies that the Newton polygon and the σ -invariant Hodge polygon of \underline{X} have a nontrivial contact point x . The specified contact point divides the Newton polygon ν into two parts ν_1 and ν_2 where the slopes of ν_1 are less than the slopes of ν_2 , and similarly divide the σ -invariant Hodge polygon μ into two parts μ_1 and μ_2 . Then \underline{X} admits a decomposition

$$\underline{X} = \underline{X}_1 \times \underline{X}_2$$

where \underline{X}_1 and \underline{X}_2 are p -divisible groups with tensors induced by a specified Levi subgroup L of G such that the Newton polygon (resp. σ -invariant Hodge-polygon) of \underline{X}_j is ν_j (resp. μ_j).

To the pair (\underline{X}, G) , we have an associated Rapoport-Zink space $\mathrm{RZ}_{X,G}$. By construction, $\mathrm{RZ}_{X,G}$ is a formal scheme over $\mathrm{Spf}(W)$, which is formally smooth and formally locally of finite type. We thus get a rigid analytic generic fibre $\mathrm{RZ}_{X,G}^{\mathrm{rig}}$, over which we construct a tower of étale covers $\mathrm{RZ}_{X,G}^\infty := \{\mathrm{RZ}_{X,G}^{K_p}\}$ where K_p runs over open compact subgroups of $G(\mathbb{Z}_p)$. This tower is equipped with a Weil descent datum over the local reflex field E and the action of $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$, where

$$J_b(\mathbb{Q}_p) = \{g \in G(K_0) : \sigma(g) = b^{-1}gb\}.$$

Hence for each integer $i > 0$, the cohomology groups

$$H^i(\mathrm{RZ}_{X,G}^{K_p}) = H_c^i(\mathrm{RZ}_{X,G}^{K_p} \otimes_{K_0} \widehat{K_0}, \mathbb{Q}_l(\dim \mathrm{RZ}_{X,G}^{K_p}))$$

fit into a tower $\{H^i(\mathrm{RZ}_{X,G}^{K_p})\}$ with a natural action of $G(\mathbb{Q}_p) \times W_E \times J_b(\mathbb{Q}_p)$, where W_E is the Weil group of the local reflex field E . For an l -adic admissible representation ρ of $J_b(\mathbb{Q}_p)$, we define a virtual representation of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$

$$H^\bullet(\mathrm{RZ}_{X,G}^\infty)_\rho := \sum_{i,j \geq 0} (-1)^{i+j} \varinjlim_{K_p} \mathrm{Ext}_{J_b(\mathbb{Q}_p)}^j(H^i(\mathrm{RZ}_{X,G}^{K_p}), \rho).$$

Now we can state our main result as follows:

Theorem 1. *Suppose that \underline{X} is of Hodge-Newton type, associated with a parabolic subgroup P and its Levi subgroup L of G . For any admissible $\bar{\mathbb{Q}}_l$ -representation ρ of $J(\mathbb{Q}_p)$, We have an equality of virtual representations of $G(\mathbb{Q}_p) \times W_E$:*

$$H^\bullet(RZ_{X,G}^\infty)_\rho = \text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H^\bullet(RZ_{X,L}^\infty)_\rho.$$

In particular, the virtual representation $H^\bullet(RZ_{X,G}^\infty)_\rho$ contains no supercuspidal representations of $G(\mathbb{Q}_p)$.

Let us point out that our proof verifies a strong form of Harris's conjecture. More precisely, our proof shows that the individual cohomology groups $H^i(RZ_{X,G}^{K_p})$ are parabolically induced, as Mantovan did in [Man08].

We briefly sketch the proof of theorem. We begin with an embedding of G into a group of EL type

$$G \hookrightarrow \tilde{G}$$

constructed in [Hong16]. Using this embedding, we construct an analogue of Rapoport-Zink space for the parabolic subgroup P from the corresponding space for the parabolic subgroup \tilde{P} of \tilde{G} such that $P = \tilde{P} \cap G$. Note that the latter space was constructed by Mantovan in [Man08].

This space, denoted by $RZ_{X,P}$, satisfies many analogous properties to the properties of Rapoport-Zink spaces. In particular, we can construct a tower $RZ_{X,P}^\infty := \{RZ_{X,P}^{K_p'}\}$ over its rigid analytic generic fibre where K_p' runs over open compact subgroups of $P(\mathbb{Z}_p)$, and define a virtual representation of $P(\mathbb{Q}_p) \times W_E$

$$H^\bullet(RZ_{X,P}^\infty)_\rho := \sum_{i,j \geq 0} (-1)^{i+j} \varinjlim_{K_p'} \text{Ext}_{J_b(\mathbb{Q}_p)}^j(H^i(RZ_{X,P}^{K_p'}), \rho)$$

from its l -adic cohomology groups.

The key point of the proof is that this space fits into a diagram

$$\begin{array}{ccc} & RZ_{X,P}^{\text{rig}} & \\ s \swarrow & & \searrow \pi_2 \\ RZ_{X,L}^{\text{rig}} & \xleftarrow{\pi_1} & RZ_{X,G}^{\text{rig}} \end{array}$$

such that

- (1) s is a closed immersion;
- (2) π_1 is a fibration in balls;
- (3) π_2 is an isomorphism.

Then we deduce the theorem by comparing the cohomologies of the spaces $RZ_{X,L}$ and $RZ_{X,G}$ with the cohomology of $RZ_{X,P}$.

We now give an overview of the structure of this work. In section 2, we introduce general notations and recall some group theoretic preliminaries. In section 3, we review

the theory of p -divisible groups with G -structure. In section 4, we review Kim's construction of Rapoport-Zink spaces for general unramified reductive groups. In section 5, after describing our assumption in detail with some results from [Hong16], we state and prove our main theorem.

Acknowledgements. I would like to express my deepest gratitude to Elena Mantovan. This study would've never been possible without her previous work for EL/PEL cases and her numerous helpful suggestions.

2. NOTATIONS AND PRELIMINARIES

2.1. General notations.

Throughout this paper, k is an algebraically closed field of positive characteristic p . We will write W for the ring of Witt vectors over k , and K_0 for its quotient field. We will let σ denote the Frobenius automorphism over k , and also its lift to W and K_0 .

Let R be a ring. For any R -module Λ and an R -algebra R' , we write $\Lambda_{R'} := \Lambda \otimes_R R'$. Similarly, if S is a scheme over $\mathrm{Spec} R$, then for any ring homomorphisms $R \rightarrow R'$ we write $S_{R'} := S \times_{\mathrm{Spec} R} \mathrm{Spec} R'$.

When R is a Noetherian ring and Λ is a free R -module of finite rank, we denote by Λ^\otimes the direct sum of all the R -modules which can be formed from Λ using the operations of taking duals, tensor products, symmetric powers and exterior powers. Note that we have a natural identification $\Lambda^\otimes \simeq (\Lambda^*)^\otimes$ where Λ^* is the dual R -module of Λ . Any isomorphism $\Lambda \xrightarrow{\sim} \Lambda'$ of free R -modules of finite rank naturally induces an isomorphism $\Lambda^\otimes \xrightarrow{\sim} (\Lambda')^\otimes$. An element of Λ^\otimes is called a *tensor* over Λ .

Let X be a p -divisible group over a \mathbb{Z}_p -scheme S . We will denote by $\mathbb{D}(X)$ the (contravariant) Dieudonné module of X , and by F the Frobenius map on $\mathbb{D}(X)$. We have a filtration $(\mathrm{Lie} X)^* \cong \mathrm{Fil}^1(\mathbb{D}(X)_S) \subset \mathbb{D}(X)_S$, called the *Hodge filtration* of X .

Let X' be another p -divisible group over S . By a *quasi-isogeny* from X to X' , we mean an element in $\mathrm{Hom}(X, X') \otimes_{\mathbb{Z}} \mathbb{Q}$. A quasi-isogeny is called an *isogeny* if it is a map of p -divisible groups.

2.2. Group theoretic preliminaries.

2.2.1. Let G be a connected reductive group over \mathbb{Q}_p . We will always assume that G is unramified, which means that the following equivalent conditions hold:

- (i) G is quasi-split and split over a finite unramified extension of \mathbb{Q}_p ;
- (ii) G admits a reductive model over \mathbb{Z}_p .

Throughout this paper, we fix the following data associated to G :

- a reductive model $G_{\mathbb{Z}_p}$ over \mathbb{Z}_p ,
- a Borel subgroup B and a maximal torus T which are both defined over \mathbb{Z}_p ,
- a closed embedding $G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(\Lambda)$ where Λ is a free \mathbb{Z}_p -module of finite rank.

We will often write $G = G_{\mathbb{Z}_p}$ if there is no risk of confusion.

Proposition 2.2.2 (cf. [Ki10], 2.3.8.). *There exists a finite family of tensors $(s_i)_{i \in I}$ on Λ such that G is the pointwise stabilizer of the s_i ; i.e., for any \mathbb{Z}_p -algebra R we have*

$$G(R) = \{g \in GL(\Lambda_R) : g((s_i)_R) = (s_i)_R \text{ for all } i \in I\}.$$

2.2.3. We let $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ denote the associated root datum, and Ω the associated Weyl group. Our choice of B determines a set of positive roots and a set of positive coroots. Recall that there is a natural action of the group Ω on $X_*(T)$ (resp. $X^*(T)$). The dominant cocharacters (resp. dominant characters) form a full set of representatives for the orbits $X_*(T)/\Omega$ (resp. $X^*(T)/\Omega$).

Let R be a \mathbb{Z}_p -algebra and $\mu : \mathbb{G}_m \rightarrow G_R$ be a cocharacter. We denote by $\{\mu\}$ the $G(R)$ -conjugacy class of μ . Using the natural identifications $\Omega \cong N_G(T)(R)/T(R)$ and $X_*(T) \cong \text{Hom}_R(\mathbb{G}_m, T_R)$, we get a bijection between $X_*(T)/\Omega$ and the set of $G(R)$ -conjugacy classes of cocharacters for G_R . When $R = W$, we have another bijection

$$\text{Hom}_W(\mathbb{G}_m, G_W)/G(W) \cong \text{Hom}_{K_0}(\mathbb{G}_m, G_{K_0})/G(K_0) \xrightarrow{\sim} G(W) \backslash G(K_0)/G(W)$$

induced by $\{\mu\} \mapsto G(W)\mu(p)G(W)$; in fact, the first bijection comes from the fact that G is split over W , whereas the second bijection is the Cartan decomposition.

2.2.4. We say that a grading $\text{gr}^\bullet(\Lambda_R)$ is *induced by μ* if the following conditions are satisfied:

- (i) the \mathbb{G}_m -action on Λ_R via μ leaves each grading stable;
- (ii) the resulting \mathbb{G}_m -action on $\text{gr}^i(\Lambda_R)$ is given by

$$\mathbb{G}_m \xrightarrow{z \mapsto z^{-i}} \mathbb{G}_m \xrightarrow{z \mapsto z \cdot \text{id}} GL(\text{gr}^i(\Lambda_R)).$$

Let S be an R -scheme, and \mathcal{E} a vector bundle on S . For a finite family of global sections (t_i) of \mathcal{E}^\otimes , we define the following scheme over S

$$\mathcal{P}_S := \mathbf{Isom}_{\mathcal{O}_S} \left([\mathcal{E}, (t_i)], [\Lambda \otimes_R \mathcal{O}_S, (s_i \otimes 1)] \right).$$

In other words, \mathcal{P}_S classifies isomorphisms of vector bundles $\mathcal{E} \cong \Lambda \otimes_R \mathcal{O}_S$ over S which match (t_i) and $(s_i \otimes 1)$.

Let $\text{Fil}^\bullet(\mathcal{E})$ be a filtration of \mathcal{E} . When \mathcal{P}_S is a trivial G -torsor, we say that $\text{Fil}^\bullet(\mathcal{E})$ is a $\{\mu\}$ -filtration with respect to (t_i) if there exists an isomorphism $\mathcal{E} \cong \Lambda \otimes_R \mathcal{O}_S$, matching (t_i) and $(1 \otimes s_i)$, which takes $\text{Fil}^\bullet(\mathcal{E})$ to a filtration of $\Lambda \otimes_R \mathcal{O}_S$ induced by $g\mu g^{-1}$ for some $g \in G(R)$. More generally, when \mathcal{P}_S a G -torsor, we say that $\text{Fil}^\bullet(\mathcal{E})$ is a $\{\mu\}$ -filtration with respect to (t_i) if it is étale-locally a $\{\mu\}$ -filtration.

2.3. Affine Deligne-Lusztig sets.

2.3.1. Consider $b \in G(K_0)$ and a cocharacter $\mu : \mathbb{G}_m \rightarrow G_W$. We denote by $B(G)$ the set of all σ -conjugacy classes in $G(K_0)$. We write $[b]_G$, or simply $[b]$ if there is no risk of confusion, for the σ -conjugacy class of $b \in G(K_0)$.

We define the *affine Deligne-Lusztig set* associated to b and $\{\mu\}$ by

$$X_{\{\mu\}}^G(b) := \{g \in G(K_0)/G(W) \mid g^{-1}b\sigma(g) \in G(W)\mu(p)G(W)\}.$$

The left multiplication by $h \in G(K_0)$ induces a bijection $X_{\{\mu\}}^G(h^{-1}b\sigma(h)) \xrightarrow{\sim} X_{\{\mu\}}^G(b)$. Moreover, the set $G(W)\mu(p)G(W)$ only depends on the conjugacy class of μ as noted

in 2.2.3. Therefore the affine Deligne-Lusztig set $X_{\{\mu\}}^G(b)$ depends only on the tuple $(G, [b], \{\mu\})$ up to bijection.

The following lemma is an immediate consequence of the definition.

Lemma 2.3.2. *Let G' be another connected reductive group over \mathbb{Z}_p .*

(1) *For $b' \in G'(K_0)$ and a cocharacter $\mu' : \mathbb{G}_m \rightarrow G'_W$, we have an isomorphism*

$$X_{\{\mu, \mu'\}}^{G \times G'}(b, b') \xrightarrow{\sim} X_{\{\mu\}}^G(b) \times X_{\{\mu'\}}^{G'}(b').$$

(2) *Let $f : G_W \rightarrow G'_W$ be a homomorphism. Then we have a natural map*

$$X_{\{\mu\}}^G(b) \longrightarrow X_{\{f \circ \mu\}}^{G'}(f(b))$$

induced by $gG(W) \mapsto f(g)G'(W)$. If f is a closed immersion, the induced map is injective.

3. p -DIVISIBLE GROUPS WITH G -STRUCTURE

We briefly review the theory of p -divisible groups with additional structures that are induced by G .

3.1. Definitions and basic notions.

3.1.1. Write $M := \Lambda_W$. Given $b \in G(K_0)$, define F to be the semi-linear map on M whose linearization is b . Then $M = \Lambda \otimes_{\mathbb{Z}_p} W$ is equipped with F -invariant tensors $(t_i) := (s_i \otimes 1)$. Let X be the p -divisible group corresponding to the Dieudonné module M with Frobenius map F . We will consider $\underline{X} := (X, (t_i))$ as a p -divisible group with additional structures encoded by the tensors (t_i) . These additional structures will be referred to as G -structure since the tensors $(t_i) = (s_i \otimes 1)$ are determined by G in the sense of Proposition 2.2.2. We will sometimes write \underline{X}_G instead of \underline{X} if we need to specify the group that induces the tensors.

One can prove that $[b]$ uniquely determines the isogeny class of \underline{X} , or equivalently the isomorphism class of F -isocrystal $M[1/p]$ with G -structure (see [RR96], 3.4.).

3.1.2. We define the *Newton set* of G by

$$\mathcal{N}(G) := (\text{Int } G(K_0) \backslash \text{Hom}_{K_0}(\mathbb{D}, G))^{\langle \sigma \rangle}$$

where \mathbb{D} is the pro-algebraic torus with character group \mathbb{Q} . With our fixed Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq G$, we can write

$$\mathcal{N}(G) = (X_*(T)_{\mathbb{Q}}/\Omega)^{\langle \sigma \rangle}.$$

In [Ko85], Kottwitz introduced the *Newton map* of G

$$\nu_G : B(G) \rightarrow \mathcal{N}(G).$$

We refer the readers to [Ko85], §4 or [RR96], §1 for its definition. The newton map induces a natural transformation of set-valued functors on the category of connected reductive groups

$$\nu : B(\cdot) \rightarrow \mathcal{N}(\cdot).$$

The element $\nu_G([b]) \in \mathcal{N}(G)$ is called the *Newton point* of \underline{X} .

3.1.3. One can show that there exists a unique $G(W)$ -conjugacy class of cocharacters $\{\mu\}$ such that the Hodge filtration of X is a $\{\mu\}$ -filtration. (see [Kim13], 2.5.8.). Alternatively, $\{\mu\}$ is uniquely determined by the condition $b \in G(W)\mu(p)G(W)$. Let $\mu \in X_*(T)$ be a unique dominant cocharacter which represents $\{\mu\}$ (see 2.2.3.). We identify μ with its image in $X_*(T)/\Omega$ and define

$$\bar{\mu} := \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\mu) \in \mathcal{N}(G),$$

for some integer m such that $\sigma^m(\mu) = \mu$. We will refer to this element as the *σ -invariant Hodge point* of \underline{X} . We will sometimes write $\bar{\mu}_G$ in order to indicate that μ is a cocharacter of G .

We have a partial order \preceq on $\mathcal{N}(G)$ defined as follows. First we define a partial order \preceq_1 on $X_*(T)_{\mathbb{R}}$ so that $\alpha \preceq_1 \alpha'$ if and only if $\alpha' - \alpha$ is a nonnegative linear combination of positive coroots. We may identify the set $X_*(T)_{\mathbb{R}}/\Omega$ with the closed Weyl chamber \bar{C} , so we get a partial order \preceq_2 on $X_*(T)_{\mathbb{R}}/\Omega$ by restricting \preceq_1 to \bar{C} . The partial order \preceq is defined as the restriction of \preceq_2 to $(X_*(T)_{\mathbb{Q}}/\Omega)^{(\sigma)}$.

With this partial order, we have a generalized Mazur's inequality

$$\nu_G([b]) \preceq \bar{\mu}.$$

We say that \underline{X} is μ -ordinary if the equality holds in the above inequality.

Example 3.1.4. (i) If $G = \mathrm{GL}_n$, we may choose (s_i) to be empty. Hence \underline{X} is simply a p -divisible group X of height n (with no additional structures).

Take T to be the diagonal torus and B to be the Borel subgroup of lower triangular matrices. Using the identification $X_*(T) \cong \mathbb{Z}^n$ we may write

$$\mathcal{N}(G) = \{(r_1, r_2, \dots, r_n) \in \mathbb{Q}^n : 0 \leq r_1 \leq r_2 \leq \dots \leq r_n\},$$

which can be regarded as a set of convex polygons with rational slopes. The partial order \preceq on $\mathcal{N}(G)$ agrees with the usual “lying above” order.

The two polygons $\nu_G([b])$ and $\bar{\mu}$ coincide with the classical Newton polygon and the Hodge polygon of X , respectively. Hence the inequality $\nu_G([b]) \preceq \bar{\mu}$ becomes the classical Mazur's inequality, and the notion of μ -ordinariness agrees with the classical notion of ordinariness.

(ii) If $G = \mathrm{GSp}_{2n}$, the tensors (t_i) encode a polarization on X .

The embedding $G \hookrightarrow \mathrm{GL}_{2n}$ induces an injective map

$$\mathcal{N}(G) \hookrightarrow \mathcal{N}(\mathrm{GL}_{2n})$$

The image of $\mathcal{N}(G)$ corresponds to the set of “symmetric polygons” in $\mathcal{N}(\mathrm{GL}_{2n})$. Under this map, we identify the two elements $\nu_G([b])$ and $\bar{\mu}$ of $\mathcal{N}(G)$ with the classical Newton polygon and the Hodge polygon of X , respectively. Then, as in (i), the inequality $\nu_G([b]) \preceq \bar{\mu}$ becomes the classical Mazur's inequality, and the notion of μ -ordinariness agrees with the classical notion of ordinariness.

(iii) Consider $G = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}_p} \mathrm{GL}_n$, the Weil restriction of GL_n , where \mathcal{O} is a finite étale extension of \mathbb{Z}_p . In this case, the tensors (t_i) encode an action of \mathcal{O} on X (or

equivalently, on M). This additional structure will be usually referred as \mathcal{O} -structure or \mathcal{O} -module structure.

Similarly to (i), we have an identification

$$\mathcal{N}(G) = \{(r_1, r_2, \dots, r_n) \in \mathbb{Q}^n : 0 \leq r_1 \leq r_2 \leq \dots \leq r_n\}.$$

The partial order \preceq on $\mathcal{N}(G)$ agrees with the usual “lying above” order.

We may regard the Newton point $\nu_G([b])$ and the σ -invariant Hodge point $\bar{\mu}$ as convex polygons with rational slopes using the above identification. The polygon $\nu_G([b])$ is closely related to the Newton polygon of X as follows: a slope λ appears in $\nu_G([b])$ with multiplicity α if and only if it appears in the Newton polygon of X with multiplicity $m\alpha$, where m is the degree of extension of \mathcal{O} over \mathbb{Z}_p . However, the same relationship generally fails to hold between the polygon $\bar{\mu}$ and the Hodge polygon of X . As a result, the notion of μ -ordinariness and the classical notion of ordinariness do not agree in general.

3.1.5. Let us now give a moduli interpretation of the affine Deligne-Lusztig set $X_{\{\mu\}}^G(b)$ in terms of quasi-isogenies of p -divisible groups with G -structure.

For $gG(W) \in X_{\{\mu\}}^G(b)$, the element $b' := g^{-1}b\sigma(g)$ in $G(K_0)$ give rise to a pair $\underline{X'} = (X', (t'_i))$ as in 3.1.1. We also obtain a quasi-isogeny $\iota : X \rightarrow X'$ corresponding to

$$\mathbb{D}(X')[1/p] \cong \Lambda \otimes_{\mathbb{Z}_p} K_0 \xrightarrow{g} \Lambda \otimes_{\mathbb{Z}_p} K_0 \cong \mathbb{D}(X)[1/p]$$

which matches (t_i) with (t'_i) . The isomorphism class of the tuple $(X, (t'_i), \iota)$ is independent of the choice of a representative g .

This association gives a bijection from $X_{\{\mu\}}^G(b)$ to the set of isomorphism classes of tuples $(X', (t'_i), \iota)$ which satisfy the following conditions:

- (i) X' is a p -divisible group over k with an isomorphism $\eta : \mathbb{D}(X') \cong \Lambda \otimes_{\mathbb{Z}_p} W$,
- (ii) (t'_i) is a family of F -invariant tensors on $\mathbb{D}(X')$ which is mapped to $(s_i \otimes 1)$ by η such that the Hodge filtration of X' is a $\{\mu\}$ -filtration,
- (iii) $\iota : X \rightarrow X'$ is a quasi-isogeny which induces an isomorphism $\mathbb{D}(X')[1/p] \xrightarrow{\sim} \mathbb{D}(X)[1/p]$ that matches (t'_i) and (t_i) .

3.2. Deformation theory for p -divisible groups with G -structure.

In this subsection, we review the construction of the “universal” deformation of p -divisible groups with G -structure, given by Faltings in [Fal99], §7. Readers may find a more detailed discussion of these results in [Mo98], §4.

3.2.1. Let R be a formally smooth W -algebra R of the form $R = W[[u_1, \dots, u_N]]$ or $R = W[[u_1, \dots, u_N]]/(p^m)$. We have a lift of Frobenius map on R , also denoted by σ , defined by $\sigma(u_i) = u_i^p$.

By a *filtered Dieudonné module* over R , we mean a 4-tuple $(\mathcal{M}, \text{Fil}^1(\mathcal{M}), \nabla, F)$ where

- \mathcal{M} is a free R -module of finite rank;
- $\text{Fil}^1(\mathcal{M}) \subset \mathcal{M}$ is a direct summand;
- $\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes \widehat{\Omega}_{R/W}$ is an integrable, topologically quasi-nilpotent connection;
- $F : \mathcal{M} \rightarrow \mathcal{M}$ is a σ -linear endomorphism,

which satisfy the following conditions:

- (i) F induces an isomorphism $(\mathcal{M} + p^{-1}\mathrm{Fil}^1(\mathcal{M})) \otimes_{R,\sigma} R \xrightarrow{\sim} \mathcal{M}$, and
- (ii) $\mathrm{Fil}^1(\mathcal{M}) \otimes_R (R/p) = \mathrm{Ker}(F \otimes \sigma_{R/p} : \mathcal{M} \otimes_R (R/p) \longrightarrow \mathcal{M} \otimes_R (R/p))$.

Proposition 3.2.2 ([Mo98], 4.1.). *There exists an anti-equivalence between the category of p -divisible groups over R and the category of filtered Dieudonné modules over R .*

3.2.3. We retain the notations from 3.1. Let \mathbf{C}_W be the category of artinian local W -algebra with residue field k . We define a *deformation* or *lifting* of X over $R \in \mathbf{C}_W$ to be a p -divisible group \mathcal{X} over R with an isomorphism $\alpha : \mathcal{X} \otimes_R k \cong X$. We also define a functor $\mathrm{Def}_X : \mathbf{C}_W \rightarrow \mathbf{Sets}$ by setting $\mathrm{Def}_X(R)$ to be the set of isomorphism classes of deformations of X over R .

Let us first assume that $G = \mathrm{GL}(\Lambda)$. Our choice of the cocharacter $\mu : \mathbb{G}_m \rightarrow \mathrm{GL}_W(M)$ determines a splitting of the Hodge filtration $\mathrm{Fil}^1(M) \subset M$. The stablizer of the complement of $\mathrm{Fil}^1(M)$ is a parabolic subgroup. We let U^μ be its unipotent radical, and R_{GL}^μ be the completed local ring of U^μ at the identity element. Then R_{GL}^μ is a formal power series ring over W , so we have a lift of Frobenius map on R_{GL}^μ as in 3.2.1.

Proposition 3.2.4 ([Fal99], §7). *Consider*

$$\mathcal{M} := M \otimes_W R_{\mathrm{GL}}^\mu, \quad \mathrm{Fil}^1(\mathcal{M}) := \mathrm{Fil}^1(M) \otimes_W R_{\mathrm{GL}}^\mu, \quad F_{\mathcal{M}} := u_t^{-1} \circ (F \otimes_W \sigma),$$

where u_t is the tautological R_{GL}^μ -point of U^μ .

- (1) *There exists a unique topologically quasi-nilpotent connection $\nabla : \mathcal{M} \longrightarrow \mathcal{M} \otimes \widehat{\Omega}_{R/W}$ which is integrable and commutes with $F_{\mathcal{M}}$.*
- (2) *If $p > 2$, the filtered Dieudonné module $(\mathcal{M}, \mathrm{Fil}^1(\mathcal{M}), \nabla, F_{\mathcal{M}})$ corresponds to the universal deformation of X via the anti-equivalence in Proposition 3.2.2.*

We will write $\mathcal{X}_{\mathrm{GL}}^\mu$ for the universal deformation of X .

3.2.5. Let us now consider the general case so that X is equipped with tensors (t_i) . Take $U_G^\mu := U^\mu \cap G_W$, which is a smooth unipotent subgroup of G_W . Let R_G^μ be the completed local ring of U_G^μ at the identity element. Then R_G^μ is a formal power series ring over W , so we get a lift of Frobenius map to R_{GL}^μ as in 3.2.1. Alternatively, we get this lift from the lift on R_{GL}^μ via the surjection $R_{\mathrm{GL}}^\mu \twoheadrightarrow R_G^\mu$.

As in Proposition 3.2.4, we have a tautological R_G^μ -point u_t of U_G^μ . We set

$$\mathcal{M}_G := M \otimes_W R_G^\mu, \quad \mathrm{Fil}^1(\mathcal{M}_G) := \mathrm{Fil}^1(M) \otimes_W R_G^\mu, \quad F_{\mathcal{M}_G} := u_t^{-1} \circ (F \otimes_W \sigma).$$

The connection $\nabla : \mathcal{M} \longrightarrow \mathcal{M} \otimes \widehat{\Omega}_{R/W}$ from Proposition 3.2.4 induces an integrable, topologically quasi-nilpotent connection $\nabla_G : \mathcal{M}_G \longrightarrow \mathcal{M}_G \otimes \widehat{\Omega}_{R/W}$, which commutes with $F_{\mathcal{M}}$ by construction. We thus have a filtered Dieudonné module over R_G^μ given by $(\mathcal{M}_G, \mathrm{Fil}^1(\mathcal{M}_G), \nabla_G, F_{\mathcal{M}_G})$.

Note that \mathcal{M}_G is equipped with a family tensors $(\mathbf{t}_i^{\mathrm{univ}}) := (t_i \otimes 1)$, whose pointwise stabilizer is isomorphic to $G_{R_G^\mu}$. It is clear from this construction that the tensors $(\mathbf{t}_i^{\mathrm{univ}})$ are $F_{\mathcal{M}}$ -invariant and lie in the 0th filtration with respect to $\mathrm{Fil}^1(\mathcal{M}_G)$.

Let \mathcal{X}_G^μ be the p -divisible group over R_G^μ which corresponds to the filtered Dieudonné module $(\mathcal{M}_G, \text{Fil}^1(\mathcal{M}_G), \nabla_G, F_{\mathcal{M}_G})$ via the anti-equivalence in Proposition 3.2.2. Alternatively, one can simply define \mathcal{X}_G^μ as the pull-back of $\mathcal{X}_{\text{GL}}^\mu$ over R_G^μ . Then \mathcal{X}_G^μ is the universal deformation of $(X, (t_i))$ in the following sense:

Proposition 3.2.6 ([Fal99], §7). *Assume that $p > 2$. Let R be a formally smooth W -algebra of the form $R = W[[u_1, \dots, u_N]]$ or $R = W[[u_1, \dots, u_N]]/(p^m)$. Let \mathcal{X} be a deformation of X over R , and consider a morphism $f : R_{\text{GL}}^\mu \rightarrow R$ induced by X via $\text{Spf } R_{\text{GL}}^\mu \cong \text{Def}_X$. Then f factors through R_G^μ if and only if the tensors (t_i) can be lifted to tensors*

$$(\mathbf{t}_i) \subset \mathbb{D}(\mathcal{X})^\otimes$$

which are Frobenius-invariant and lie in the 0th filtration with respect to the Hodge filtration. If this holds, then we necessarily have $(f^ \mathbf{t}_i^{\text{univ}}) = (\mathbf{t}_i)$.*

We let $\text{Def}_{X,G}$ denote the image of the closed immersion $\text{Spf } (R_G^\mu) \hookrightarrow \text{Def}_X$. Proposition 3.2.6 says that this subscheme classifies deformations of $\underline{X} = (X, (t_i))$ over formal power series rings over W of $W/(p^m)$. Moreover, our definition of $\text{Def}_{X,G}$ is independent of the choice of (t_i) and $\mu \in \{\mu\}$; the independence of the choice of (t_i) is evident from construction, and the independence of the choice of μ follows from the universal property described in Proposition 3.2.6.

Proposition 3.2.7. *Let G' be a reductive group over \mathbb{Z}_p , and let $b' \in G'(K_0)$ which gives rise to a p -divisible group X' with tensors (t'_i) in the sense of 3.1.1.*

- (1) *The natural morphism $\text{Def}_X \times \text{Def}_{X'} \rightarrow \text{Def}_{X \times X'}$, defined by taking the product of deformations, induces an isomorphism*

$$\text{Def}_{X,G} \times \text{Def}_{X',G'} \xrightarrow{\sim} \text{Def}_{X \times X', G \times G'}.$$

- (2) *Let $f : G_W \rightarrow G'_W$ be a homomorphism over W such that $f(b) = b'$. Then the morphism*

$$\text{Def}_{X,G} \longrightarrow \text{Def}_{X',G'},$$

corresponding to the induced map $U_G^\mu \rightarrow U_{G'}^{f \circ \mu}$, depends only on f . Furthermore, this morphism is injective if f is a closed immersion.

In particular, the deformation space $\text{Def}_{X,G}$ depends only on the pair (G, b) .

4. RAPOPORT-ZINK SPACES OF HODGE TYPE

In this section, we discuss the construction and key properties of Rapoport-Zink spaces of Hodge type, following [Kim13]. We retain the notations from the previous section.

4.1. Construction.

4.1.1. Let us first describe the original construction of Rapoport-Zink spaces in [RZ96].

Take $G = \mathrm{GL}(\Lambda)$. Let Nilp_W denote the category of W -algebra where p is nilpotent. We define the covariant functor $\mathrm{RZ}_X : \mathrm{Nilp}_W \rightarrow \mathbf{Sets}$ by setting $\mathrm{RZ}_X(R)$ to be the set of isomorphism classes of pairs (\mathcal{X}, ι) where

- \mathcal{X} is a p -divisible group over R ,
- $\iota : X_{R/p} \rightarrow \mathcal{X}_{R/p}$ is a quasi-isogeny.

The functor RZ_X depends only on $[b]$ up to isomorphism.

Rapoport and Zink proved that the functor RZ_X can be represented by a formal scheme which is locally formally of finite type and formally smooth over W . We also let RZ_X denote the representing formal scheme. We denote by $\mathcal{X}_{\mathrm{GL},b}$ the universal p -divisible group over RZ_X .

4.1.2. We now consider the general case so that X is equipped with tensors (t_i) on M .

Consider $(\mathcal{X}, \iota) \in \mathrm{RZ}_X(R)$ with $R \in \mathrm{Nilp}_W$. Then ι induces an isomorphism

$$\mathbb{D}(\iota) : \mathbb{D}(\mathcal{X}_{R/p})[1/p] \xrightarrow{\sim} \mathbb{D}(X_{R/p})[1/p].$$

Using this isomorphism, we may identify (\hat{t}_i) as a family of tensors on $\mathbb{D}(\mathcal{X}_{R/p})[1/p]$. This family can be uniquely lifted to a family of tensors (\hat{t}_i) on $\mathbb{D}(\mathcal{X})[1/p]$.

Proposition 4.1.3 ([Kim13], Theorem 4.9.1.). *Assume that $p > 2$. Then there exists a closed formal subscheme $\mathrm{RZ}_{X,G} \subset \mathrm{RZ}_X$, which is formally smooth over W , with the following universal property: Let R be a formally smooth and formally finitely generated algebra over either W or W/p^m for some m . Consider a morphism $f : \mathrm{Spf}(R) \rightarrow \mathrm{RZ}_X$ and a p -divisible group \mathcal{X} over $\mathrm{Spec}(R)$ which pulls back to $f^*\mathcal{X}_{\mathrm{GL},b}$ over $\mathrm{Spf}(R)$. Then f factors through $\mathrm{RZ}_{X,G}$ if and only if there exists a (unique) family of crystalline Tate tensors (\mathbf{t}_i) on $\mathbb{D}(\mathcal{X})$ such that*

- (1) *for some ideal of definition J of R containing p , the pull-back of (\mathbf{t}_i) over R/J agrees with the pull-back of (\hat{t}_i) over R/J ;*
- (2) *for a p -adic lift \mathcal{R} of R which is formally smooth over W , the \mathcal{R} -scheme*

$$\mathcal{P}_{\mathcal{R}} := \mathbf{Isom}_{\mathcal{R}}\left([\mathbb{D}(\mathcal{X})_{\mathcal{R}}, (\mathbf{t}_i)_{\mathcal{R}}], [\Lambda \otimes_{\mathbb{Z}_p} \mathcal{R}, (s_i \otimes 1)]\right),$$

defined as in 2.2.4, is a G -torsor;

- (3) *The Hodge filtration of \mathcal{X} is a $\{\mu\}$ -filtration with respect to (\mathbf{t}_i) .*

Moreover, the closed formal subscheme $\mathrm{RZ}_{X,G} \subset \mathrm{RZ}_X$ is independent of the choice of (s_i) .

We obtain the “universal p -divisible group” $\mathcal{X}_{G,b}$ over $\mathrm{RZ}_{X,G}$ by taking the pull-back of $\mathcal{X}_{\mathrm{GL},b}$. Applying the universal property to an open affine covering of $\mathcal{X}_{\mathrm{GL},b}$, we also obtain a family of “universal tensors” $(\mathbf{t}_i^{\mathrm{univ}})$ on $\mathbb{D}(\mathcal{X}_{G,b})$ (see [Kim13], 4.7.1.).

Example 4.1.4. Let us consider the setting of Example 3.1.4.(iii), i.e., $G = \mathrm{Res}_{\mathcal{O}|\mathbb{Z}_p} \mathrm{GL}_n$ for a finite étale extension \mathcal{O} of \mathbb{Z}_p . In this setting, our construction of $\mathrm{RZ}_{X,G}$ is compatible with the construction of Rapoport-Zink spaces of EL type in [RZ96]. In other words, for any $R \in \mathrm{Nilp}_W$, the set $\mathrm{RZ}_{X,G}(R)$ classifies the isomorphism classes of pairs (\mathcal{X}, ι) where

- \mathcal{X} is a p -divisible group over R , endowed with an action of \mathcal{O} such that
$$\det_R(a, \text{Lie}(\mathcal{X})) = \det(a, \text{Fil}^0(X)_{K_0}) \quad \text{for all } a \in \mathcal{O},$$
- $\iota : X_{R/p} \rightarrow \mathcal{X}_{R/p}$ is a quasi-isogeny, commuting with the action of \mathcal{O} .

4.2. Functorialities and other structures.

4.2.1. From now on, we will always assume $p > 2$ to ensure that $\text{RZ}_{X,G}$ exists.

The moduli interpretation of $X_{\{\mu\}}^G(b)$ in 3.1.5 gives a natural isomorphism

$$X_{\{\mu\}}^G(b) \cong \text{RZ}_{X,G}(k).$$

For $x \in \text{RZ}_{X,G}(k)$, we write $(X_x, (t_{x,i}), \iota_x)$ for the triple given by moduli interpretation of $X_{\{\mu\}}^G(b)$ in 3.1.5. As explained in [Kim13], 4.8, we have a natural isomorphism

$$\text{Def}_{X_x,G} \cong (\widehat{\text{RZ}_{X,G}})_x.$$

Proposition 4.2.2 ([Kim13], Theorem 4.9.1.). *Let G' be a reductive group over \mathbb{Z}_p , and let $b' \in G'(K_0)$ which gives rise to a p -divisible group X' with tensors (t'_i) in the sense of 3.1.1. We choose a cocharacter $\mu' : \mathbb{G}_m \rightarrow G'_W$ associated to $(X', (t'_i))$, as described in 3.1.3.*

- (1) *The natural morphism $\text{RZ}_X \times_{\text{Spf}(W)} \text{RZ}_{X'} \rightarrow \text{RZ}_{X \times X'}$, defined by the product of p -divisible groups with quasi-isogeny, induces an isomorphism*

$$\text{RZ}_{X,G} \times_{\text{Spf}(W)} \text{RZ}_{X',G'} \xrightarrow{\sim} \text{RZ}_{X \times X', G \times G'}$$

such that the induced product decompositions

$$X_{\{\mu, \mu'\}}^{G \times G'}(b, b') \xrightarrow{\sim} X_{\{\mu\}}^G(b) \times X_{\{\mu'\}}^{G'}(b'),$$

$$\text{Def}_{X \times X', G \times G'} \xrightarrow{\sim} \text{Def}_{X,G} \times \text{Def}_{X',G'},$$

obtained via the isomorphisms described in 4.2.1, are compatible with the product decompositions in Lemma 2.3.2 and Proposition 3.2.7.

- (2) *Let $f : G_W \rightarrow G'_W$ be a homomorphism over W such that $f(b) = b'$. Then there exists a unique morphism*

$$\text{RZ}_{X,G} \longrightarrow \text{RZ}_{X',G'}$$

which induces the following Cartesian diagrams:

$$\begin{array}{ccc} X_{\{\mu\}}^G(b) & \longrightarrow & X_{\{\mu'\}}^{G'}(b') \\ \wr \downarrow & & \wr \downarrow \\ \text{RZ}_{X,G}(k) & \longrightarrow & \text{RZ}_{X',G'}(k) \end{array} \quad \begin{array}{ccc} \text{Def}_{X_x,G} & \longrightarrow & \text{Def}_{X_{x'},G'} \\ \wr \downarrow & & \wr \downarrow \\ (\widehat{\text{RZ}_{X,G}})_x & \longrightarrow & (\widehat{\text{RZ}_{X',G'}})_{x'} \end{array}$$

for any $x \in \text{RZ}_{X,G}(k)$ and its image $x' \in \text{RZ}_{X',G'}(k)$. Here the vertical maps are the isomorphisms described in 4.2.1, and the top maps are as in Lemma 2.3.2 and Proposition 3.2.7.

Furthermore, if f is a closed immersion, the associated map $\text{RZ}_{X,G} \longrightarrow \text{RZ}_{X',G'}$ is a closed immersion.

4.2.3. For the rest of this section, we assume that $k = \bar{\mathbb{F}}_p$.

We define the group valued functor J_b on the category of \mathbb{Q}_p -algebra by

$$J_b(R) := \{g \in G(R \otimes_{\mathbb{Q}_p} K_0) : gb\sigma(g)^{-1} = b\}$$

for any \mathbb{Q}_p -algebra R . Note that $J_b(\mathbb{Q}_p)$ can be identified with the group of quasi-isogenies of X that preserve the tensors (t_i) . Hence one can show that $\mathrm{RZ}_{X,G}$ carries a natural left $J_b(\mathbb{Q}_p)$ -action defined by

$$\gamma(\mathcal{X}, \iota) = (\mathcal{X}, \iota \circ \gamma^{-1})$$

for any $R \in \mathrm{Nilp}_W$, $(\mathcal{X}, \iota) \in \mathrm{RZ}_{X,G}(R)$ and $\gamma \in J_b(\mathbb{Q}_p)$.

4.2.4. Let E be the field of definition of the $G(K_0)$ -conjugacy class of μ . Note that E is a finite unramified extension of \mathbb{Q}_p . Let d be the degree of the extension, and write τ for the Frobenius automorphism of K_0 relative to E .

For any formal scheme S over $\mathrm{Spf}(W)$, we write $S^\tau := S \times_{\mathrm{Spf}(W), \tau} \mathrm{Spf}(W)$. For any $R \in \mathrm{Nilp}_W$, we define R^τ to be R viewed as a W -algebra via τ . Note that we have $S^\tau(R) = S(R^\tau)$.

A *Weil descent datum* on S over \mathcal{O}_E is an isomorphism $\Phi : S \xrightarrow{\sim} S^\tau$. If there exists a formal scheme S_0 over $\mathrm{Spf}(\mathcal{O}_E)$ with $(S_0)_W \cong S$, then there exists a natural Weil descent datum over \mathcal{O}_E on S . Such a Weil descent datum is called *effective*.

For any $(\mathcal{X}, \iota) \in \mathrm{RZ}_X(R)$ with $R \in \mathrm{Nilp}_W$, we define $(\mathcal{X}^\Phi, \iota^\Phi) \in \mathrm{RZ}_X(R^\tau)$ as follows:

- \mathcal{X}^Φ is \mathcal{X} viewed as a p -divisible group over R^τ ,
- ι^Φ is the quasi-isogeny

$$\iota^\Phi : X_{R^\tau/p} = (\tau^* X)_{R/p} \xrightarrow{\mathrm{Frob}^{-d}} X_{R/p} \xrightarrow{\iota} \mathcal{X}_{R/p} = \mathcal{X}_{R/p}^\Phi$$

where $\mathrm{Frob}^d : X \rightarrow \tau^* X$ is the relative q -Frobenius with $q = p^d$.

The association $(\mathcal{X}, \iota) \mapsto (\mathcal{X}^\Phi, \iota^\Phi)$ defines a Weil descent datum $\Phi : \mathrm{RZ}_X \xrightarrow{\sim} \mathrm{RZ}_X^\tau$ over \mathcal{O}_E . One can check that Φ restricts to a Weil descent datum $\Phi : \mathrm{RZ}_{X,G} \xrightarrow{\sim} \mathrm{RZ}_{X,G}^\tau$ over \mathcal{O}_E by looking at k -points and the formal completions thereof. The Weil descent datum Φ clearly commutes with the $J_b(\mathbb{Q}_p)$ -action.

Remark. In the setting of Example 4.1.4, i.e., $G = \mathrm{Res}_{\mathcal{O}|\mathbb{Z}_p} \mathrm{GL}_n$ for a finite étale extension \mathcal{O} of \mathbb{Z}_p , one easily checks that the $J_b(\mathbb{Q}_p)$ -action and the Weil descent datum described above are compatible with the ones defined by Rapoport and Zink in [RZ96].

4.3. Rigid analytic tower of Rapoport-Zink spaces.

4.3.1. We continue to assume that $p > 2$ and $k = \bar{\mathbb{F}}_p$. Since $\mathrm{RZ}_{X,G}$ is locally formally of finite type over $\mathrm{Spf}(W)$, we have a rigid analytic generic fibre $\mathrm{RZ}_{X,G}^{\mathrm{rig}}$ (see [Ber96]). The $J_b(\mathbb{Q}_p)$ -action and the Weil descent datum on $\mathrm{RZ}_{X,G}$ naturally lifts to $\mathrm{RZ}_{X,G}^{\mathrm{rig}}$. (For an analytic space \mathcal{S} over K_0 , we define a Weil descent datum on \mathcal{S} over E to be an isomorphism $\Phi : \mathcal{S} \xrightarrow{\sim} \mathcal{S}^\tau$.)

Recall that $\mathcal{X}_{G,b}$ over $\mathrm{RZ}_{X,G}$ is the universal p -divisible group over $\mathrm{RZ}_{X,G}$, endowed with “universal tensors” $(\mathbf{t}_i^{\mathrm{univ}})$ on $\mathbb{D}(\mathcal{X}_{G,b})$. These universal tensors induce tensors $(\mathbf{t}_{i,\mathrm{ét}}^{\mathrm{univ}})$ on the Tate module $T_p(\mathcal{X}_{G,b})$ as explained in [Kim13], 7.1.6.

For any open compact subgroup K_p of $G(\mathbb{Z}_p)$, we define the following rigid analytic étale cover of $\mathrm{RZ}_{X,G}^{\mathrm{rig}}$:

$$\mathrm{RZ}_{X,G}^{K_p} := \mathbf{Isom}_{\mathrm{RZ}_{X,G}^{\mathrm{rig}}} \left([\Lambda, (s_i)], [T_p(\mathcal{X}_{G,b}), (\mathbf{t}_{i,\mathrm{ét}}^{\mathrm{univ}})] \right) / K_p.$$

The $J_b(\mathbb{Q}_p)$ -action and the Weil descent datum over E on $\mathrm{RZ}_{X,G}^{\mathrm{rig}}$ pull back to $\mathrm{RZ}_{X,G}^{K_p}$. As the level K_p varies, these covers form a tower $\{\mathrm{RZ}_{X,G}^{K_p}\}$ with Galois group $G(\mathbb{Z}_p)$. We will usually denote this tower by $\mathrm{RZ}_{X,G}^\infty$.

Proposition 4.3.2 ([Kim13], Proposition 7.4.8.). *There exists a right $G(\mathbb{Q}_p)$ -action on the tower $\mathrm{RZ}_{X,G}^\infty = \{\mathrm{RZ}_{X,G}^{K_p}\}$ extending the Galois action of $G(\mathbb{Z}_p)$, which commutes with the natural $J_b(\mathbb{Q}_p)$ -action and the Weil descent datum over E .*

4.3.3. We fix a prime $l \neq p$, and let W_E denote the Weil group of E . For any level $K_p \subset G(\mathbb{Z}_p)$, we consider the cohomology groups

$$H^i(\mathrm{RZ}_{X,G}^{K_p}) = H_c^i(\mathrm{RZ}_{X,G}^{K_p} \otimes_{K_0} \widehat{K_0}, \mathbb{Q}_l(\dim \mathrm{RZ}_{X,G}^{K_p}))$$

where $\widehat{K_0}$ is the completion of an algebraic closure of K_0 . As the level K_p varies, these cohomology groups form a tower $\{H^i(\mathrm{RZ}_{X,G}^{K_p})\}$ for each i . Proposition 4.3.2 shows that we have a natural action of $G(\mathbb{Q}_p) \times W_E \times J_b(\mathbb{Q}_p)$ on these towers.

Let ρ be an admissible l -adic representation of $J_b(\mathbb{Q}_p)$. We will consider the groups

$$H^{i,j}(\mathrm{RZ}_{X,G}^\infty)_\rho := \varinjlim_{K_p} \mathrm{Ext}_{J_b(\mathbb{Q}_p)}^j(H^i(\mathrm{RZ}_{X,G}^{K_p}), \rho).$$

One can check that these groups satisfy the following properties (cf. [Man08], Theorem 8.):

- (1) The groups $H^{i,j}(\mathrm{RZ}_{X,G}^\infty)_\rho := \varinjlim_{K_p} \mathrm{Ext}_{J_b(\mathbb{Q}_p)}^j(H^i(\mathrm{RZ}_{X,G}^{K_p}), \rho)$ vanishes for almost all $i, j \geq 0$.
- (2) There is a natural action of $G(\mathbb{Q}_p) \times W_E$ on $H^{i,j}(\mathrm{RZ}_{X,G}^\infty)_\rho$.
- (3) The representations $H^{i,j}(\mathrm{RZ}_{X,G}^\infty)_\rho$ are admissible.

Hence we can define

$$H^\bullet(\mathrm{RZ}_{X,G}^\infty)_\rho := \sum_{i,j \geq 0} (-1)^{i+j} H^{i,j}(\mathrm{RZ}_{X,G}^\infty)_\rho$$

as a virtual representation of $G(\mathbb{Q}_p) \times W_E$.

5. HARRIS CONJECTURE FOR RAPOPORT-ZINK SPACES OF HODGE TYPE

In this section, we state and prove a variant of Harris' conjecture under the assumption that \underline{X} is of Hodge-Newton type (see 5.1.2 for definition).

We retain the assumptions and notations from §4. In particular, we assume that $p > 2$ and $k = \overline{\mathbb{F}}_p$, and choose a prime $l \neq p$.

5.1. Hodge-Newton filtration for p -divisible groups with G -structure.

We recall some of the main results in [Hong16].

5.1.1. As described in [Hong16], 4.1.1, we choose an embedding

$$(5.1.1.1) \quad G \hookrightarrow \tilde{G}$$

such that \tilde{G} is a group of EL type and $\bar{\mu}_G = \bar{\mu}_{\tilde{G}}$. For simplicity, we assume that $\tilde{G} = \text{Res}_{\mathcal{O}|\mathbb{Z}_p} \text{GL}_n$ for some finite étale extension \mathcal{O} of \mathbb{Z}_p . (In general, \tilde{G} is a product of the groups of this form.)

Regarding b as an element in $\tilde{G}(K_0)$ via the embedding (5.1.1.1), we obtain \mathcal{O} -structure on X (see Example 3.1.4. (iii)). We will write \tilde{X} for X equipped with \mathcal{O} -structure. Following [Hong16], 4.1.4, we will refer to \tilde{X} as the *EL realization* of \underline{X} .

The Newton point and the σ -invariant Hodge point of \tilde{X} can be regarded as convex polygons with rational slope via the following identification (see Example 3.1.4. (iii)):

$$\mathcal{N}(\tilde{G}) = \{(r_1, r_2, \dots, r_n) \in \mathbb{Q}^n : 0 \leq r_1 \leq r_2 \leq \dots \leq r_n\}.$$

We define the *Newton polygon* of \underline{X} , denoted by $\nu_{\underline{X}}$, to be the polygon corresponding to the Newton point of \tilde{X} . We similarly define the *σ -invariant Hodge polygon* of \underline{X} , which we denote by $\bar{\mu}_{\underline{X}}$.

5.1.2. In [Ko97], Kottwitz defined the map

$$\kappa_G : B(G) \rightarrow \pi_1(G)_{\langle \sigma \rangle},$$

called the Kottwitz map of G (see [Ko97], §4 and §7 for its definition). We write $\mu^\natural := \kappa_G(b)$. One can identify μ^\natural with the image of μ under the natural projection $X_*(T) \twoheadrightarrow \pi_1(G)_{\langle \sigma \rangle}$.

We say that \underline{X} is of *Hodge-Newton type* if there exists a proper Levi subgroup L of G containing T , such that $b \in L(K_0)$ and $\kappa_L(b) = \mu^\natural$. If \underline{X} is of Hodge-Newton type, $\nu_{\underline{X}}$ and $\bar{\mu}_{\underline{X}}$ have nontrivial contact points which are break points of $\nu_{\underline{X}}$ (see [Hong16], Proposition 4.2.1.).

For the rest of this paper, \underline{X} is always assumed to be of Hodge-Newton type. Let P be the parabolic subgroup of G associated to L . We choose a Borel pair (\tilde{B}, \tilde{T}) of \tilde{G} such that $T \subseteq \tilde{T}$ and $B \subseteq \tilde{B}$. Then we get a parabolic subgroup \tilde{P} the corresponding Levi subgroup \tilde{L} of $\tilde{G} = \text{Res}_{\mathcal{O}|\mathbb{Z}_p} \text{GL}_n$ such that $P = \tilde{P} \cap G$ and $L = \tilde{L} \cap G$. Note that \tilde{L} is of the form

$$\tilde{L} = \text{Res}_{\mathcal{O}|\mathbb{Z}_p} \text{GL}_{j_1} \times \text{Res}_{\mathcal{O}|\mathbb{Z}_p} \text{GL}_{j_2} \times \dots \times \text{Res}_{\mathcal{O}|\mathbb{Z}_p} \text{GL}_{j_r}.$$

We write \tilde{L}_j for the j th factor of this decomposition, and take L_j to be the image of L under the projection $\tilde{L} \twoheadrightarrow \tilde{L}_j$. Let $b_j \in L_j(K_0)$ be the image of b under the map $L \twoheadrightarrow L_j$. We also choose a decomposition of Λ

$$\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \dots \oplus \Lambda_r$$

induced by the above decomposition of \tilde{L} .

For simplicity, we assume that our Levi subgroup gives only one contact point $x = (x_1, x_2)$. Let ν_1 denote the polygon consisting of the first x_1 slopes of $\nu_{\underline{X}}$ and ν_2 the polygon consisting of the remaining ones. We similarly define $\bar{\mu}_1$ and $\bar{\mu}_2$ for $\bar{\mu}_{\underline{X}}$. Then by [Hong16], Theorem 4.2.3, we have a decomposition

$$(5.1.2.1) \quad \underline{X} = \underline{X}_1 \times \underline{X}_2$$

where \underline{X}_j is a p -divisible group with L_j -structure, induced by b_j , with the Newton polygon ν_j and the σ -invariant Hodge polygon $\bar{\mu}_j$.

The decomposition (5.1.2.1) is called the *Hodge-Newton decomposition* of \underline{X} (associated to the Levi subgroup L). The induced filtration

$$(5.1.2.2) \quad 0 \subset \underline{X}_1 \subset \underline{X}$$

is called the *Hodge-Newton filtration* of \underline{X} (associated to the Levi subgroup L).

The existence of the Hodge-Newton decomposition (5.1.2.1) gives maps on the affine Deligne-Lusztig sets

$$(5.1.2.3) \quad X_{\{\mu\}}^G(b) \xrightarrow{\sim} X_{\{\mu\}}^L(b) \hookrightarrow X_{\{\mu_1\}}^{L_1}(b_1) \times X_{\{\mu_2\}}^{L_2}(b_2).$$

For details, see the remark after [Hong16], Theorem 4.2.3.

Proposition 5.1.3 (cf. [Hong16], Theorem 4.2.4.). *Let $\underline{\mathcal{X}} = (\mathcal{X}, (t_i))$ be a p -divisible group over $R \in \text{Nilp}_W$ with Tate tensors which lifts \underline{X} . More precisely, \mathcal{X} is a lift of X over R equipped with tensors (t_i) which lift (t_i) . Then there exists a unique filtration*

$$0 \subset \underline{\mathcal{X}}_1 \subset \underline{\mathcal{X}}$$

which lifts the Hodge-Newton filtration (5.1.2.2). In other words, there exists a short exact sequence of p -divisible groups over R

$$0 \longrightarrow \mathcal{X}_1 \longrightarrow \mathcal{X} \longrightarrow \mathcal{X}_2 \longrightarrow 0$$

such that \mathcal{X}_j lifts X_j and is equipped with tensors $(t_{j,i})$ which lift $(t_{j,i})$.

Proof. Using the embedding (5.1.1.1), we obtain \mathcal{O} -structure on \mathcal{X} which lifts the \mathcal{O} -structure on \tilde{X} . Let $\tilde{\mathcal{X}}$ denote \mathcal{X} with \mathcal{O} -structure. Since \tilde{X} is of Hodge-Newton type, [Sh13], Theorem 5.4 gives a short exact sequence

$$0 \longrightarrow \tilde{\mathcal{X}}_1 \longrightarrow \tilde{\mathcal{X}} \longrightarrow \tilde{\mathcal{X}}_2 \longrightarrow 0$$

where $\tilde{\mathcal{X}}_j$ is a p -divisible group over W with \mathcal{O} -structure which lifts the EL realization \tilde{X}_j of \underline{X}_j . Setting $\mathcal{M}_j := \mathbb{D}(\mathcal{X}_j)$, we have a short exact sequence of Dieudonné modules

$$0 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}_1 \longrightarrow 0$$

where \mathcal{M}_j lifts M_j as a \mathcal{O} -module.

It remains to find tensors on \mathcal{M}_j which lift $(t_{j,i})$. We choose a basis \mathcal{B}_j for \mathcal{M}_j as a \mathcal{O} -module, and let B_j be the corresponding basis for M_j . We write tensors $(t_{j,i})$ in terms of the elements in B_j . By considering the corresponding expressions in terms of \mathcal{B}_j , we obtain tensors $(t_{j,i})$ which lift $(t_{j,i})$. \square

5.2. Harris conjecture: statement.

5.2.1. We continue to assume that \underline{X} is of Hodge-Newton type. We retain the notations from 5.1. In particular, we have a parabolic subgroup P and a Levi subgroup L of G .

Let ρ be an admissible l -adic representation of $J_b(\mathbb{Q}_p)$. In 4.3.3, we defined a virtual representation of $G(\mathbb{Q}_p) \times W_E$

$$H^\bullet(\mathrm{RZ}_{X,G}^\infty)_\rho := \sum_{i,j \geq 0} (-1)^{i+j} H^{i,j}(\mathrm{RZ}_{X,G}^\infty)_\rho.$$

Note that, since $b \in L(K_0)$ by the assumption that \underline{X} is of Hodge-Newton type, we can define L -structure on X induced by b . From the associated Rapoport-Zink space $\mathrm{RZ}_{X,L}$, we obtain a virtual representation of $L(\mathbb{Q}_p) \times W_E$

$$H^\bullet(\mathrm{RZ}_{X,L}^\infty)_\rho := \sum_{i,j \geq 0} (-1)^{i+j} H^{i,j}(\mathrm{RZ}_{X,L}^\infty)_\rho.$$

We will consider the above group as a virtual representation of $P(\mathbb{Q}_p) \times W_E$ by letting the unipotent radical of $P(\mathbb{Q}_p)$ act trivially.

We can now state our main theorem as follows:

Theorem 5.2.2. *For any admissible $\bar{\mathbb{Q}}_l$ -representation ρ of $J(\mathbb{Q}_p)$, We have an equality of virtual representations of $G(\mathbb{Q}_p) \times W_E$:*

$$H^\bullet(\mathrm{RZ}_{X,G}^\infty)_\rho = \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H^\bullet(\mathrm{RZ}_{X,L}^\infty)_\rho.$$

In particular, the virtual representation $H^\bullet(\mathrm{RZ}_{X,G}^\infty)_\rho$ contains no supercuspidal representations of $G(\mathbb{Q}_p)$.

5.3. Rigid analytic tower associated to the parabolic subgroup.

For our proof of Theorem 5.2.2, we construct an intermediate tower of rigid analytic spaces associated to the parabolic subgroup P .

5.3.1. Following Mantovan in [Man08], Definition 9, we define the functor $\mathrm{RZ}_{X,\bar{P}} : \mathrm{Nilp}_W \rightarrow \mathbf{Sets}$ by setting $\mathrm{RZ}_{X,\bar{P}}(R)$ to be the set of isomorphism classes of triples $(\mathcal{X}, 0 \subset \mathcal{X}_1 \subset \mathcal{X}, \iota)$ where

- \mathcal{X} is a p -divisible group over R , endowed with an action of \mathcal{O} ,
- $0 \subset \mathcal{X}_1 \subset \mathcal{X}$ is a filtration of \mathcal{X} by p -divisible subgroups over R , with p -divisible group subquotients, which is preserved by the action of \mathcal{O} ,
- $\iota : X_{R/p} \rightarrow \mathcal{X}_{R/p}$ is a quasi-isogeny, compatible with the action of \mathcal{O} , which induces a quasi-isogeny $\iota_1 : (X_1)_{R/p} \rightarrow (\mathcal{X}_1)_{R/p}$

such that for all $a \in \mathcal{O}$,

$$\begin{aligned} \det_R(a, \mathrm{Lie}(\mathcal{X})) &= \det(a, \mathrm{Fil}^0(X)_{K_0}), \\ \det_R(a, \mathrm{Lie}(\mathcal{X}_1)) &= \det(a, \mathrm{Fil}^0(X_1)_{K_0}). \end{aligned}$$

We will use the following identification of $\mathrm{RZ}_{X,\bar{P}}$ as a subfunctor of $\mathrm{RZ}_{X,\bar{G}} \times \mathrm{RZ}_{X,\bar{L}_1}$: for any $R \in \mathrm{Nilp}_W$, a tuple $(\mathcal{X}, \iota, \mathcal{X}_1, \iota_1) \in \mathrm{RZ}_{X,\bar{G}}(R) \times \mathrm{RZ}_{X,\bar{L}_1}(R)$ is in $\mathrm{RZ}_{X,\bar{P}}(R)$ if and only if the map

$$(\mathcal{X}_1)_{R/p} \xrightarrow{\iota_1^{-1}} (X_1)_{R/p} \hookrightarrow X_{R/p} \xrightarrow{\iota} \mathcal{X}_{R/p}$$

is an injective morphism.

Using this identification, one can show that $\mathrm{RZ}_{X,\tilde{P}}$ is represented by a formal scheme which is locally formally of finite type and formally smooth over W (see [Man08], Proposition 11.). We let $\mathrm{RZ}_{X,\tilde{P}}$ also denote the representing formal scheme, and let $\mathrm{RZ}_{X,\tilde{P}}^{\mathrm{rig}}$ be its rigid analytic generic fibre. By [Sh13], Proposition 6.3, the map

$$\tilde{\pi}_2 : \mathrm{RZ}_{X,\tilde{P}} \xrightarrow{\sim} \mathrm{RZ}_{X,\tilde{G}},$$

defined by $(\mathcal{X}, \iota, \mathcal{X}_1, \iota_1) \mapsto (\mathcal{X}, \iota)$, gives an isomorphism of the rigid analytic generic fibres.

Proposition 5.3.2. *There exists a closed formal subscheme $\mathrm{RZ}_{X,P} \subset \mathrm{RZ}_{X,\tilde{P}}$, which is formally smooth over W , such that for any $R \in \mathrm{Nilp}_W$,*

$$\mathrm{RZ}_{X,P}(R) = \{(\mathcal{X}, \iota, \mathcal{X}_1, \iota_1) \in \mathrm{RZ}_{X,\tilde{P}}(R) : (\mathcal{X}, \iota) \in \mathrm{RZ}_{X,G} \text{ and } (\mathcal{X}_1, \iota_1) \in \mathrm{RZ}_{X,L_1}\}.$$

Proof. By the functoriality of Rapoport-Zink spaces described in Proposition 4.2.2, the embedding $G \hookrightarrow \tilde{G}$ induces a closed embedding

$$\mathrm{RZ}_{X,G} \hookrightarrow \mathrm{RZ}_{X,\tilde{G}}.$$

We define $\mathrm{RZ}_{X,P} := \mathrm{RZ}_{X,\tilde{P}} \times_{\mathrm{RZ}_{X,\tilde{G}}} \mathrm{RZ}_{X,G}$. Then we have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{RZ}_{X,P} & \xrightarrow{\pi_2} & \mathrm{RZ}_{X,G} \\ \downarrow & & \downarrow \\ \mathrm{RZ}_{X,\tilde{P}} & \xrightarrow{\tilde{\pi}_2} & \mathrm{RZ}_{X,\tilde{G}} \end{array}$$

Now the assertion follows from Proposition 5.1.3. □

5.3.3. We describe the universal property of $\mathrm{RZ}_{X,P} \subset \mathrm{RZ}_{X,\tilde{P}}$ in an analogous way to the universal property of $\mathrm{RZ}_{X,G} \subset \mathrm{RZ}_X$ in Proposition 4.1.3.

Recall that we chose a decomposition of the \mathbb{Z}_p -module $\Lambda = \Lambda_1 \oplus \Lambda_2$ in 5.1.2, which induces a filtration $0 \subset \Lambda_1 \subset \Lambda$. We consider the universal filtered p -divisible group $0 \subset \mathcal{X}_{\tilde{L}_1,b} \subset \mathcal{X}_{\tilde{P},b}$ over $\mathrm{RZ}_{X,\tilde{P}}$.

Let R be a formally smooth and formally finitely generated algebra over either W or W/p^m for some m . For a morphism $f : \mathrm{Spf}(R) \rightarrow \mathrm{RZ}_{X,\tilde{P}}$, let $0 \subset \mathcal{X}_1 \subset \mathcal{X}$ be a filtered p -divisible groups over $\mathrm{Spec}(R)$ which pulls back to $0 \subset f^*\mathcal{X}_{\tilde{L}_1,b} \subset f^*\mathcal{X}_{\tilde{P},b}$ over $\mathrm{Spf}(R)$. We choose tensors (\hat{t}_i) on $\mathbb{D}(\mathcal{X})[1/p]$ and $(\hat{t}_{1,i})$ on $\mathbb{D}(\mathcal{X}_1)[1/p]$ as in 4.1.2. Then f factors through $\mathrm{RZ}_{X,G}$ if and only if there exists a (unique) family of crystalline Tate tensors (\mathbf{t}_i) on $\mathbb{D}(\mathcal{X})$ and $(\mathbf{t}_{1,i})$ on $\mathbb{D}(\mathcal{X}_1)$ such that

- (1) for some ideal of definition J of R containing p , the pull-back of (\mathbf{t}_i) (resp. $(\mathbf{t}_{1,i})$) over R/J agrees with the pull-back of (\hat{t}_i) (resp. $(\hat{t}_{1,i})$) over R/J ;

(2) for a p -adic lift \mathcal{R} of R which is formally smooth over W , the \mathcal{R} -scheme

$$\mathcal{P}'_{\mathcal{R}} := \mathbf{Isom}_{\mathcal{R}} \left([\mathbb{D}(\mathcal{X}_1)_{\mathcal{R}}, (\mathbf{t}_{1,i})_{\mathcal{R}}] \subset [\mathbb{D}(\mathcal{X})_{\mathcal{R}}, (\mathbf{t}_i)_{\mathcal{R}}], [(\Lambda_1)_{\mathcal{R}}, (s_{1,i} \otimes 1)] \subset [\Lambda_{\mathcal{R}}, (s_i \otimes 1)] \right)$$

is a P -torsor;

(3) The Hodge filtration of \mathcal{X} (resp. \mathcal{X}_1) is a $\{\mu\}$ -filtration with respect to (\mathbf{t}_i) (resp. $(\mathbf{t}_{1,i})$).

Here the \mathcal{R} -scheme $\mathcal{P}'_{\mathcal{R}}$ in (2) classifies the isomorphisms $\mathbb{D}(\mathcal{X})_{\mathcal{R}} \cong \Lambda_{\mathcal{R}}$ which match the filtrations $0 \subset \mathbb{D}(\mathcal{X}_1)_{\mathcal{R}} \subset \mathbb{D}(\mathcal{X})_{\mathcal{R}}$ and $0 \subset (\Lambda_1)_{\mathcal{R}} \subset \Lambda_{\mathcal{R}}$ and map the tensors $(\mathbf{t}_{1,i})$ and (\mathbf{t}_i) to $(s_{1,i} \otimes 1)$ and $(s_i \otimes 1)$.

We obtain the “universal filtered p -divisible group” $0 \subset \mathcal{X}_{L_1,b} \subset \mathcal{X}_{P,b}$ over $\mathrm{RZ}_{X,P}$ by taking the pull-back of the filtered p -divisible group $0 \subset \mathcal{X}_{\tilde{L}_1,b} \subset \mathcal{X}_{\tilde{P},b}$. Applying the universal property to an “open affine covering” of this filtration, we also obtain “universal tensors” $(\mathbf{t}_{1,i}^{\mathrm{univ},P})$ on $\mathbb{D}(\mathcal{X}_{L_1,b})$ and $(\mathbf{t}_i^{\mathrm{univ},P})$ on $\mathbb{D}(\mathcal{X}_{P,b})$. These universal tensors induce tensors $(\mathbf{t}_{1,i,\mathrm{\acute{e}t}}^{\mathrm{univ},P})$ and $(\mathbf{t}_{i,\mathrm{\acute{e}t}}^{\mathrm{univ},P})$ on the Tate modules $T_p(\mathcal{X}_{L_1,b})$ and $T_p(\mathcal{X}_{P,b})$, respectively, in the sense of [Kim13], 7.1.6.

5.3.4. The formal scheme $\mathrm{RZ}_{X,P}$ is formally smooth by construction. Let $\mathrm{RZ}_{X,\tilde{P}}^{\mathrm{rig}}$ be its rigid analytic generic fibre. Note that the map $\pi_2 : \mathrm{RZ}_{X,P} \rightarrow \mathrm{RZ}_{X,G}$ in the proof of Proposition 5.3.2 gives an isomorphism on the rigid analytic generic fibre. Hence we have a $J_b(\mathbb{Q}_p)$ -action and a Weil descent datum over E on $\mathrm{RZ}_{X,P}^{\mathrm{rig}}$ induced by the ones on $\mathrm{RZ}_{X,G}^{\mathrm{rig}}$.

For any open compact subgroup K_p' of $P(\mathbb{Z}_p)$, we define the following rigid analytic étale cover of $\mathrm{RZ}_{X,P}^{\mathrm{rig}}$:

$$\mathrm{RZ}_{X,P}^{K_p'} := \mathbf{Isom}_{\mathrm{RZ}_{X,P}^{\mathrm{rig}}} \left([\Lambda_1, (s_{1,i})] \subset [\Lambda, (s_i)], [T_p(\mathcal{X}_{L_1,b}), (\mathbf{t}_{1,i,\mathrm{\acute{e}t}}^{\mathrm{univ},P})] \subset [T_p(\mathcal{X}_{P,b}), (\mathbf{t}_{i,\mathrm{\acute{e}t}}^{\mathrm{univ},P})] \right) / K_p'.$$

The $J_b(\mathbb{Q}_p)$ -action and the Weil descent datum over E on $\mathrm{RZ}_{X,P}^{\mathrm{rig}}$ pull back to $\mathrm{RZ}_{X,P}^{K_p'}$. As the level K_p varies, these covers form a tower $\{\mathrm{RZ}_{X,P}^{K_p'}\}$ with Galois group $P(\mathbb{Z}_p)$. We will usually write $\mathrm{RZ}_{X,P}^{\infty}$ for this tower.

Similarly to the tower $\mathrm{RZ}_{X,G}^{\infty}$, the new tower $\mathrm{RZ}_{X,P}^{\infty}$ is endowed with a natural $P(\mathbb{Q}_p)$ action which commutes with the $J_b(\mathbb{Q}_p)$ -action and the Weil descent datum over E . Hence, as in 4.3.3, the cohomology groups

$$H^i(\mathrm{RZ}_{X,P}^{K_p'}) = H_c^i(\mathrm{RZ}_{X,P}^{K_p} \otimes_{K_0} \widehat{K_0}, \mathbb{Q}_l(\dim \mathrm{RZ}_{X,P}^{K_p'}))$$

form a tower $\{H^i(\mathrm{RZ}_{X,P}^{K_p'})\}$ for each i , endowed with a natural action of $G(\mathbb{Q}_p) \times W_E \times J_b(\mathbb{Q}_p)$.

Let ρ be an admissible l -adic representation of $J_b(\mathbb{Q}_p)$. As in the case of $\mathrm{RZ}_{X,G}^{\infty}$, the groups

$$H^{i,j}(\mathrm{RZ}_{X,P}^{\infty})_{\rho} := \varinjlim_{K_p'} \mathrm{Ext}_{J_b(\mathbb{Q}_p)}^j(H^i(\mathrm{RZ}_{X,P}^{K_p'}), \rho).$$

satisfy the following properties:

- (1) The groups $H^{i,j}(\mathrm{RZ}_{X,P}^\infty)_\rho := \varinjlim_{K_p'} \mathrm{Ext}_{J_b(\mathbb{Q}_p)}^j(H^i(\mathrm{RZ}_{X,P}^{K_p'}), \rho)$ vanishes for almost all $i, j \geq 0$.
- (2) There is a natural action of $P(\mathbb{Q}_p) \times W_E$ on $H^{i,j}(\mathrm{RZ}_{X,P}^\infty)_\rho$.
- (3) The representations $H^{i,j}(\mathrm{RZ}_{X,P}^\infty)_\rho$ are admissible.

Hence we can define

$$H^\bullet(\mathrm{RZ}_{X,P}^\infty)_\rho := \sum_{i,j \geq 0} (-1)^{i+j} H^{i,j}(\mathrm{RZ}_{X,P}^\infty)_\rho$$

as a virtual representation of $P(\mathbb{Q}_p) \times W_E$.

5.4. Harris conjecture: proof.

Lemma 5.4.1 (cf. [Sh13], Proposition 6.1.). *The three rigid analytic spaces $\mathrm{RZ}_{X,L}^{\mathrm{rig}}$, $\mathrm{RZ}_{X,P}^{\mathrm{rig}}$ and $\mathrm{RZ}_{X,G}^{\mathrm{rig}}$ fit into a diagram*

$$\begin{array}{ccc} & \mathrm{RZ}_{X,P}^{\mathrm{rig}} & \\ s \swarrow & & \searrow \pi_2 \\ \mathrm{RZ}_{X,L}^{\mathrm{rig}} & \xleftarrow{\pi_1} & \mathrm{RZ}_{X,G}^{\mathrm{rig}} \end{array}$$

such that

- (1) s is a closed immersion;
- (2) π_1 is a fibration in balls;
- (3) π_2 is an isomorphism.

Proof. Note that (3) follows immediately from the construction of π_2 in the proof of Proposition 5.3.2.

Consider the map

$$\tilde{s} : \mathrm{RZ}_{X,\tilde{L}} \cong \mathrm{RZ}_{X_1,\tilde{L}_1} \times \mathrm{RZ}_{X_2,\tilde{L}_2} \longrightarrow \mathrm{RZ}_{X,\tilde{P}}$$

defined by $(\mathcal{X}_1, \iota_1, \mathcal{X}_2, \iota_2) \mapsto (\mathcal{X}_1 \oplus \mathcal{X}_2, \iota_1 \oplus \iota_2, \mathcal{X}_1, \iota_1)$ on the R -points for any $R \in \mathrm{Nilp}_W$. Define s to be the restriction of \tilde{s} on $\mathrm{RZ}_{X,L}$. Since $\tilde{\pi}_2 \circ \tilde{s}$ is simply the closed immersion $\mathrm{RZ}_{X,\tilde{L}} \hookrightarrow \mathrm{RZ}_{X,G}$, s is also a closed immersion which factors through the embedding $\mathrm{RZ}_{X,P} \hookrightarrow \mathrm{RZ}_{X,\tilde{P}}$. Hence we prove (1).

In order to construct π_1 , we consider the map

$$\tilde{\pi}_1 : \mathrm{RZ}_{X,\tilde{P}} \longrightarrow \mathrm{RZ}_{X_1,\tilde{L}_1} \times \mathrm{RZ}_{X_2,\tilde{L}_2} \xrightarrow{\sim} \mathrm{RZ}_{X,\tilde{L}}$$

defined by $(\mathcal{X}, \iota, \mathcal{X}_1, \iota_1) \mapsto (\mathcal{X}_1, \iota_1, \mathcal{X}/\mathcal{X}_1, \iota_2) \mapsto (\mathcal{X}_1 \oplus (\mathcal{X}/\mathcal{X}_1), \iota_1 \oplus \iota_2)$ on the R -points, where $\iota_2 : (\mathcal{X}/\mathcal{X}_1)_{R/p} \longrightarrow (X_2)_{R/p}$ is a quasi-isogeny induced by ι and ι_2 . We define π_1 be the restriction of $\tilde{\pi}_1$ on $\mathrm{RZ}_{X,P}$.

We claim that π_1 factor through the embedding $\mathrm{RZ}_{X,L} \hookrightarrow \mathrm{RZ}_{X,\tilde{L}}$. It suffices to show that $\pi_2^{-1} \circ \pi_1$ factors through $\mathrm{RZ}_{X,L} \hookrightarrow \mathrm{RZ}_{X,\tilde{L}}$. We only need to check this on the set of k -points and the completions thereof. On the k -points, this is an immediate consequence of the existence of the Hodge-Newton decomposition; indeed, the map is given by (5.1.2.3). On the completion at $(\mathcal{X}, \iota) \in \mathrm{RZ}_{X,G}(k)$, we get a map $\mathrm{Def}_{X,G} \longrightarrow$

$\mathrm{Def}_{X_1, L_1} \times \mathrm{Def}_{X_2, L_2} \cong \mathrm{Def}_{X_1 \times X_2, L_1 \times L_2}$ given by $\mathcal{X} \mapsto \mathcal{X}_1 \times (\mathcal{X}/\mathcal{X}_1)$. Since $\mathbb{D}(\mathcal{X}_1)$ and $\mathbb{D}(\mathcal{X}/\mathcal{X}_1)$ lift $\mathbb{D}(X_1)$ and $\mathbb{D}(X_2)$ respectively, we can clearly lift the tensors on $\mathbb{D}(X) \cong \mathbb{D}(X_1) \oplus \mathbb{D}(X_2)$ which defines L -structure to $\mathbb{D}(\mathcal{X}_1) \oplus \mathbb{D}(\mathcal{X}/\mathcal{X}_1)$. This implies that $\mathcal{X}_1 \times (\mathcal{X}/\mathcal{X}_1) \in \mathrm{Def}_{X, L}$, establishing the claim.

Finally, one easily sees that π_1 is a fibration in balls. In fact, for any $(\mathcal{X}, \iota) \in \mathrm{RZ}_{X, L}(k)$, the completion of $\mathrm{RZ}_{X, P}$ at $s(\mathcal{X}, \iota)$ is isomorphic to $\mathrm{Def}_{X, G}$, which is isomorphic to $\mathrm{Spf} R_G^\mu$ as described in 3.2.5. \square

Proposition 5.4.2. *For any admissible l -adic representation ρ of $J_b(\mathbb{Q}_p)$, we have*

$$H^\bullet(\mathrm{RZ}_{X, L}^\infty)_\rho = H^\bullet(\mathrm{RZ}_{X, P}^\infty)_\rho$$

as virtual representations of $P(\mathbb{Q}_p) \times W_E$.

Proof. For open compact subgroups $K_p \subset P(\mathbb{Q}_p)$, we get morphisms of rigid analytic spaces

$$s_{K_p'} : \mathrm{RZ}_{X, L}^{K_p' \cap L(\mathbb{Q}_p)} \longrightarrow \mathrm{RZ}_{X, P}^{K_p'} \quad \text{and} \quad \pi_{1, K_p'} : \mathrm{RZ}_{X, P}^{K_p'} \longrightarrow \mathrm{RZ}_{X, L}^{K_p' \cap L(\mathbb{Q}_p)}$$

which are $J_b(\mathbb{Q}_p) \times P(\mathbb{Q}_p)$ -equivariant and compatible with the Weil descent datum. Moreover, $s_{K_p'}$'s are closed immersions and satisfy $\pi_{1, K_p'} \circ s_{K_p'} = \mathrm{id}_{\mathrm{RZ}_{X, L}^{K_p' \cap L(\mathbb{Q}_p)}}$.

Recall that $0 \subset \mathcal{X}_{\tilde{L}, b} \subset \mathcal{X}_{\tilde{P}, b}$ is the universal filtered p -divisible group over $\mathrm{RZ}_{X, \tilde{P}}$. In [Man08], Mantovan introduces a formal scheme $\mathrm{RZ}_{X, \tilde{P}}^{(m)} \longrightarrow \mathrm{RZ}_{X, \tilde{P}}$ for each integer $m > 0$ such that, for any morphism of formal schemes $f : S \longrightarrow \mathrm{RZ}_{X, \tilde{P}}$, the filtered p^m -torsion subgroup $0 \subset f^* \mathcal{X}_{L, b}[p^m] \subset f^* \mathcal{X}_{P, b}[p^m]$ is split if and only if f factors through $\mathrm{RZ}_{X, \tilde{P}}^{(m)} \longrightarrow \mathrm{RZ}_{X, \tilde{P}}$. Pulling back $\mathrm{RZ}_{X, \tilde{P}}^{(m)}$ over $\mathrm{RZ}_{X, P}$, we obtain a formal scheme $\mathrm{RZ}_{X, P}^{(m)} \longrightarrow \mathrm{RZ}_{X, P}$ with an analogous universal property. Then $\mathrm{RZ}_{X, P}^{(m)}$ and $\mathrm{RZ}_{X, P}$ are isomorphic as formal schemes over $\mathrm{RZ}_{X, L}$, as $\mathrm{RZ}_{X, \tilde{P}}^{(m)}$ and $\mathrm{RZ}_{X, \tilde{P}}$ are isomorphic as formal schemes over $\mathrm{RZ}_{X, \tilde{L}}$ (see [Sh13], §6.). We write $\mathrm{RZ}_{X, P}^{(m), \mathrm{rig}}$ for the rigid analytic generic fibre of $\mathrm{RZ}_{X, P}^{(m)}$.

Let $K_p'^{(m)} := \ker(P(\mathbb{Z}_p) \rightarrow P(\mathbb{Z}_p/p^m \mathbb{Z}_p))$. We define two distinct covers $\mathcal{P}_m \longrightarrow \mathrm{RZ}_{X, P}^{(m)}$ and $\mathcal{P}'_m \longrightarrow \mathrm{RZ}_{X, P}^{(m)}$ by the Cartesian diagrams

$$\begin{array}{ccc} \mathcal{P}_m & \longrightarrow & \mathrm{RZ}_{X, P}^{(m), \mathrm{rig}} \\ \downarrow & & \downarrow \\ \mathrm{RZ}_{X, P}^{K_p'^{(m)}} & \longrightarrow & \mathrm{RZ}_{X, P}^{\mathrm{rig}} \end{array} \quad \begin{array}{ccc} \mathcal{P}'_m & \longrightarrow & \mathrm{RZ}_{X, P}^{(m), \mathrm{rig}} \\ \downarrow & & \downarrow \pi_1 \\ \mathrm{RZ}_{X, L}^{K_p'^{(m)}} & \longrightarrow & \mathrm{RZ}_{X, L}^{\mathrm{rig}} \end{array}$$

Since π_1 is a fibration in balls, one obtains a quasi-isomorphism

$$R\Gamma_c(\mathrm{RZ}_{X, L}^{K_p'^{(m)}} \times \mathbb{C}_p, \bar{\mathbb{Q}}_l) \cong R\Gamma_c(\mathrm{RZ}_{X, L}^{K_p'^{(m)}} \times \mathbb{C}_p, \bar{\mathbb{Q}}_l(-D))[-2D]$$

where $D = \dim \mathrm{RZ}_{X,P} - \dim \mathrm{RZ}_{X,L}$. Moreover, one can argue as in [Man08], Lemma 31 and Proposition 32 to deduce a quasi-isomorphism

$$R\Gamma_c(\mathrm{RZ}_{X,P}^{K_p'^{(m)}} \times \mathbb{C}_p, \bar{\mathbb{Q}}_l) \cong R\Gamma_c(\mathrm{RZ}_{X,L}^{K_p'^{(m)}} \times \mathbb{C}_p, \bar{\mathbb{Q}}_l).$$

Thus we have quasi-isomorphisms

$$R\Gamma_c(\mathrm{RZ}_{X,P}^{K_p'^{(m)}} \times \mathbb{C}_p, \bar{\mathbb{Q}}_l) \cong R\Gamma_c(\mathrm{RZ}_{X,L}^{K_p'^{(m)}} \times \mathbb{C}_p, \bar{\mathbb{Q}}_l(-D))[-2D] \quad \text{for all } m > 0,$$

thereby obtaining the desired identity. \square

Proposition 5.4.3. *For any admissible l -adic representation ρ of $J_b(\mathbb{Q}_p)$, we have*

$$H^\bullet(\mathrm{RZ}_{X,G}^\infty)_\rho = \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H^\bullet(\mathrm{RZ}_{X,P}^\infty)_\rho$$

as virtual representations of $P(\mathbb{Q}_p) \times W_E$.

Proof. For any open compact subgroup $K_p \subseteq G(\mathbb{Z}_p)$, we have natural morphisms of rigid analytic spaces

$$\pi_{2,K_p} : \mathrm{RZ}_{X,P}^{K_p \cap P(\mathbb{Q}_p)} \rightarrow \mathrm{RZ}_{X,G}^{K_p}$$

which are $J_b(\mathbb{Q}_p) \times P(\mathbb{Q}_p)$ -equivariant and compatible with the Weil descent datum. These maps are closed immersions, so one finds

$$\mathrm{RZ}_{X,G}^{K_p} \cong \mathrm{RZ}_{X,G}^{K_p} \times_{\mathrm{RZ}_{X,G}^{\mathrm{rig}}} \mathrm{RZ}_{X,P}^{\mathrm{rig}} \cong \coprod_{K_p \backslash G(\mathbb{Q}_p)/P(\mathbb{Q}_p)} \mathrm{RZ}_{X,P}^{K_p \cap P(\mathbb{Q}_p)},$$

which yields the desired equality. \square

Proposition 5.4.2 and 5.4.3 together imply Theorem 5.2.2.

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